8 The Plane Separation Axiom

Definition (convex set of points) Let $\{\mathcal{S},\mathcal{L},d\}$ be a metric geometry and let $\mathcal{S}_1 \subseteq \mathcal{S}$. $\mathcal{S}_1$ is said to be convex if for every two points $P,Q \in \mathcal{S}$, the segment $\overline{PQ}$ is a subset of $\mathcal{S}_1$.

1. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are convex subsets of a metric geometry, prove that $\mathcal{S}_1 \cap \mathcal{S}_2$ is convex.

2. If ℓ is a line in a metric geometry, prove that ℓ is convex.

3. Consider the set of ordered pairs (x,y) with $(x-1)^2+y^2=9$, $0<x<4$ and $0<y$. Explain is this set (and when it is) convex.

Definition (plane separation axiom (PSA), half planes)
A metric geometry $\{\mathcal{S},\mathcal{L},d\}$ satisfies the plane separation axiom (PSA) if for every $\ell \in \mathcal{L}$ there are two subsets H_1 and H_2 of $\mathcal{S}$ (called half planes determined by ℓ) such that
(i) $\mathcal{S}-\ell = H_1 \cup H_2$;
(ii) H_1 and H_2 are disjoint and each is convex;
(iii) If $A \in H_1$ and $B \in H_2$ then $\overline{AB} \cap \ell \neq \emptyset$.

Theorem Let ℓ be a line in a metric geometry. If both H_1, H_2 and H_1', H_2' satisfy the conditions of PSA for the line ℓ then either $H_1 = H_1'$ (and $H_2 = H_2'$) or $H_1 = H_2'$ (and $H_2 = H_1'$).

4. Prove the above theorem.

Definition (lie on the same side, lie on opposite sides, side that contains)
Let $\{\mathcal{S},\mathcal{L},d\}$ be a metric geometry which satisfies PSA, let $\ell \in \mathcal{L}$, and let H_1 and H_2 be the half planes determined by ℓ. Two points A and B lie on the same side of ℓ if they both belong to H_1 or both belong to H_2. Points A and B lie on opposite sides of ℓ if one belongs to H_1 and one belongs to H_2. If $A \in H_1$, we say that H_1 is the side of ℓ that contains A.

Theorem Let $\{\mathcal{S},\mathcal{L},d\}$ be a metric geometry which satisfies PSA. Let A and B be two points of $\mathcal{S}$ not on a given line ℓ. Then
(i) A and B are on opposite sides of ℓ if and only if $\overline{AB} \cap \ell \neq \emptyset$.
(ii) A and B are on the same side of ℓ if and only if either $A=B$ or $\overline{AB} \cap \ell = \emptyset$.

5. Prove the above theorem.

6. Let ℓ be a line in a metric geometry which satisfies PSA. If P and Q are on opposite sides of ℓ and if Q and R are on opposite sides of ℓ then P and R are on the same side of ℓ.

7. Let ℓ be a line in a metric geometry which satisfies PSA. If P and Q are on opposite sides of ℓ and if Q and R are on the same side of ℓ then P and R are on opposite sides of ℓ.

Theorem Let ℓ be a line in a metric geometry with PSA. Assume that H_1 is a half plane determined by the line ℓ. If H_1 is also a half plane determined by the line ℓ', then $\ell = \ell'$.

8. Prove the above theorem.

Definition (edge) If H_1 is a half plane determined by the line ℓ, then the edge of H_1 is ℓ.

9. Determine are the statements true or false:
(a) If A,B are points, then $\overline{AB}$ is a convex set.
(b) If A,B are points, then $\{A,B\}$ is a convex set.
(c) The intersection of two convex sets is a convex set. (d) The union of two convex sets is a convex set. (e) $\overline{BC} = \overleftrightarrow{BC} \cap \triangle ABC$.

10. Let ℓ be a line in a metric geometry $\{\mathcal{S},\mathcal{L},d\}$ which satisfies PSA. We write $P \sim Q$ if P and Q are on the same side of ℓ. Prove that $\sim$ is an equivalence relation on $\mathcal{S}-\ell$. How many equivalence classes are there and what are they?

11. Consider the distance function d_N defined on the Euclidean plane as follows: Let every line other than L_0 have the usual Euclidean ruler, and for the line L_0, let the ruler be $f: L_0 \to \mathbb{R}$ where

$$f((0,y)) = \begin{cases} y, & \text{if } y \text{ is not an integer,} \\ -y, & \text{if } y \text{ is an integer.} \end{cases}$$

(You may assume that this function is a ruler.)

(a) Show that $\{(0,y) \mid \frac{1}{2} \le y \le \frac{3}{2}\}$ is a convex set in $(\mathbb{R}^2, \mathcal{L}_E, d_E)$, the Euclidean plane with the usual Euclidean distance, but not in $(\mathbb{R}^2, \mathcal{L}_E, d_N)$, the Euclidean plane with the new distance.

(b) Find the line segment from $(0, \frac{1}{2})$ to $(0, \frac{3}{2})$ in $(\mathbb{R}^2, \mathcal{L}_E, d_N)$. Show that it is a convex set in $(\mathbb{R}^2, \mathcal{L}_E, d_N)$ but not in $(\mathbb{R}^2, \mathcal{L}_E, d_E)$.

(c) Show that $(\mathbb{R}^2, \mathcal{L}_E, d_N)$, the Euclidean plane with this new distance d_N, does not satisfy PSA, the Plane Separation Axiom.

9 PSA for the Euclidean and Poincaré Planes

Notation **($X^\perp$ or X perp)** If $X = (x,y) \in \mathbb{R}^2$ then $X^\perp$ (read "X perp") is the element $X^\perp = (-y, x) \in \mathbb{R}^2$.

Lemma

(a) If $X \in \mathbb{R}^2$ then $\langle X, X^\perp \rangle = 0$.

(b) If $X \in \mathbb{R}^2$ and $X \neq (0,0)$ then $\langle Z, X^\perp \rangle = 0$ implies that $Z = tX$ for some $t \in \mathbb{R}$.

1. Prove the above lemma.

Proposition If P and Q are distinct points in $\mathbb{R}^2$ then

$$\overleftrightarrow{PQ} = \{A \in \mathbb{R}^2 \mid \langle A - P, (Q-P)^\perp = 0\}.$$

2. Prove the above proposition.

Definition **(Euclidean half planes)**

Let $\ell = \overleftrightarrow{PQ}$ be a Euclidean line. The Euclidean half planes determined by ℓ are

$$H^+ = \{A \in \mathbb{R}^2 \mid \langle A - P, (Q-P)^\perp \rangle > 0\}.$$

$$H^- = \{A \in \mathbb{R}^2 \mid \langle A - P, (Q-P)^\perp \rangle < 0\}.$$

Proposition The Euclidean half planes determined by $\ell = \overleftrightarrow{PQ}$ are convex.

3. Prove the above proposition.

Proposition The Euclidean Plane satisfies PSA.

4. Prove the above proposition.

Definition **(Poincaré half planes)**

If $\ell = {}_aL$ is a type I line in the Poincaré Plane then the Poincaré half planes determined by ℓ are

$$H_+ = \{(x,y) \in \mathbb{H} \mid x > a\}, \qquad H_- = \{(x,y) \in \mathbb{H} \mid x < a\}. \tag{2}$$

If $\ell = {}_cL_r$, is a type II line then the Poincaré half planes determine by ℓ are

$$H_+ = \{(x,y) \in \mathbb{H} \mid (x-c)^2 + y^2 > r^2\}, \qquad H_- = \{(x,y) \in \mathbb{H} \mid (x-c)^2 + y^2 < r^2\}.$$

Proposition The Poincaré Plane satisfies PSA.

5. Prove the above proposition.

6. Prove that the Euclidean half plane H^- is convex.

7. Let ℓ be a line in the Euclidean Plane and suppose that $A \in H^+$ and $B \in H^-$. Show that $\overline{AB} \cap \ell \neq \emptyset$ in the following way. Let

$$g(t) = \langle A + t(B-A) - P, (Q-P)^\perp \rangle \text{ if } t \in \mathbb{R}.$$

Evaluate $g(0)$ and $g(1)$, show that g is continuous, and then prove that $\overline{AB} \cap \ell \neq \emptyset$.

8. If $\ell = {}_aL$ is a type I line in the Poincaré Plane then prove that

a. H_+ and H_- as defined in Equation (2) are convex.

b. If $A \in H_+$ and $B \in H_-$ then $\overline{AB} \cap \ell \neq \emptyset$.

9. For the Taxicab Plane $(\mathbb{R}^2, \mathcal{L}_E, d_T)$ prove that

a. If $A = (x_1, y_1)$, $B = (x_2, y_2)$ and $C = (x_3, y_3)$ are collinear but do not lie on a vertical line then $A - B - C$ if and only if $x_1 * x_2 * x_3$.

b. The Taxicab Plane satisfies PSA.

Aksiom razdvajanja ravni

Definicija (konveksan skup)

Neka je $\{\mathcal{S}, \mathcal{L}, d\}$ metrična geometrija i neka je $\mathcal{S}_1 \subseteq \mathcal{S}$. Za skup tački $\mathcal{S}_1$ kažemo da je konveksan ako za svake dvije tačke $P, Q \in \mathcal{S}_1$, duž $\overline{PQ}$ je podskup od $\mathcal{S}_1$.

(#) Ako su $\mathcal{S}_1$ i $\mathcal{S}_2$ konveksni podskupovi metričke geometrije, dokazati da je $\mathcal{S}_1 \cap \mathcal{S}_2$ konveksan.

Rj.

Izaberimo dvije proizvoljne tačke $A, B \in \mathcal{S}_1 \cap \mathcal{S}_2$ takve da $A \neq B$.

$$A, B \in \mathcal{S}_1 \cap \mathcal{S}_2 \Rightarrow \underbrace{\begin{array}{ccc} A, B \in \mathcal{S}_1 & \text{i} & A, B \in \mathcal{S}_2 \\ \Downarrow \ \mathcal{S}_1 \text{ konv} & & \Downarrow \ \mathcal{S}_2 \text{ konv} \\ \overline{AB} \subseteq \mathcal{S}_1 & & \overline{AB} \subseteq \mathcal{S}_2 \end{array}}$$

$$\Downarrow$$

$$\overline{AB} \subseteq \mathcal{S}_1 \cap \mathcal{S}_2$$

Kako su A i B bile dvije proizvoljne tačke iz $\mathcal{S}_1 \cap \mathcal{S}_2$ slijedi da je $\mathcal{S}_1 \cap \mathcal{S}_2$ konveksan skup.

(#) Ako je ℓ prava u metričnoj geometriji, dokazati da je ℓ konveksan skup.

Rj. Zadatak ćemo riješiti na dva načina.

I način

Neka je ℓ prava u metričnoj geometriji.

Trebamo pokazati da $\forall A, B \in \ell \quad \overline{AB} \subseteq \ell$

Prisjetimo se definicije metrične geometrije:

- geometrija incidencije $\{\mathcal{S}, \mathcal{L}\}$ zajedno sa fjom udalj. d
- fja udalj. d zadovolj. postulat mjere ($\forall \ell \in \mathcal{L}$ $\exists$ mjera)

Geometrija incidencije:
- abstraktna geom. $\{\mathcal{S}, \mathcal{L}\}$
- $\forall$ dvije razl. tačke iz $\mathcal{S}$ $\exists!$ prava
- $\exists$ tri nekolin. tačke

$A, B \in \ell \overset{\text{geom. incid.}}{\Longrightarrow} \ell$ je jedinstvena prava koja sadrži tačke A i B.

$$\overline{AB} = \{M \in \mathcal{S} \mid A-M-B \text{ ili } M=A \text{ ili } M=B\} =$$
$$= \{M \in \ell \mid A-M-B \text{ ili } M=A \text{ ili } M=B\} \subseteq \ell$$

II način

Pretpostavimo suprotno tvrdnji tj. pretpostavimo $\exists A, B \in \ell$ t.d.
$\overline{AB} \not\subseteq \ell \Rightarrow \exists C \in \overline{AB} = \{M \in \mathcal{S} \mid A-M-B \text{ ili } M=A \text{ ili } M=B\}$
t.d. $C \notin \ell$

$C \in \overline{AB} \Rightarrow A, B, C$ su kolin. tačke $\Rightarrow \exists p \;\; A, B, C \in p$

Za A, B $\exists! \ell$ t.d. $A, B \in \ell \overset{A,B \in p}{\Longrightarrow} \ell = p \Rightarrow C \in \ell$

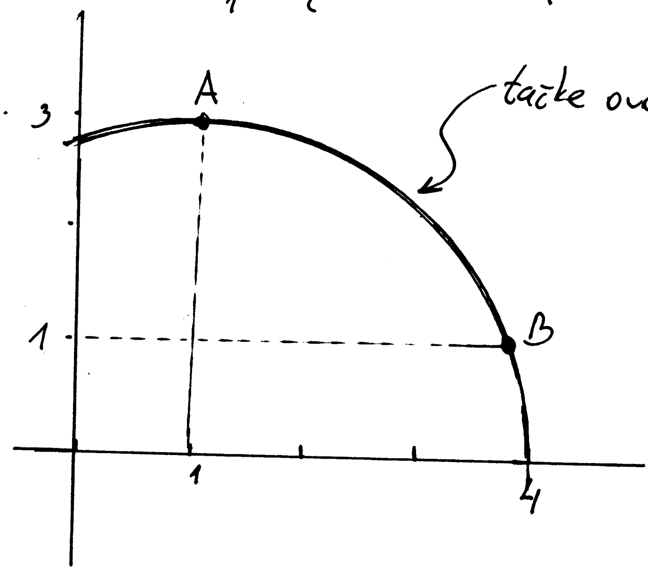
#kontradikcija

Pretpostavka suprotna tvrdnji nas vodi u kontradikciju pa nije tačna.

(#) Posmatrajmo skup uređenih parova (x,y) sa osobinom da je $(x-1)^2+y^2=9$, $0<x<4$ i $0<y$. Diskutovati da li je ovo (i kada je) konveksan skup.

Rj: Kao podskup Euklidove ravni ovo nije konveksan skup.

$$\mathcal{S}_1 = \{(x,y)\in\mathbb{R}^2 \mid (x-1)^2+y^2=9,\ 0<x<4 \text{ i } 0<y\}$$



tačke ovog luka su u $\mathcal{S}_1$

$A(1,3)\in\mathcal{S}_1$

$B(2\sqrt{2}+1,\ 1)\in\mathcal{S}_1$

ali $\overline{AB}\not\subseteq\mathcal{S}_1$

Kao podskup Poincaré-ove ravni ovo jest konveksan skup.

<u>Definicija</u> (aksiom razdvajanja ravni (PSA), poluravni)

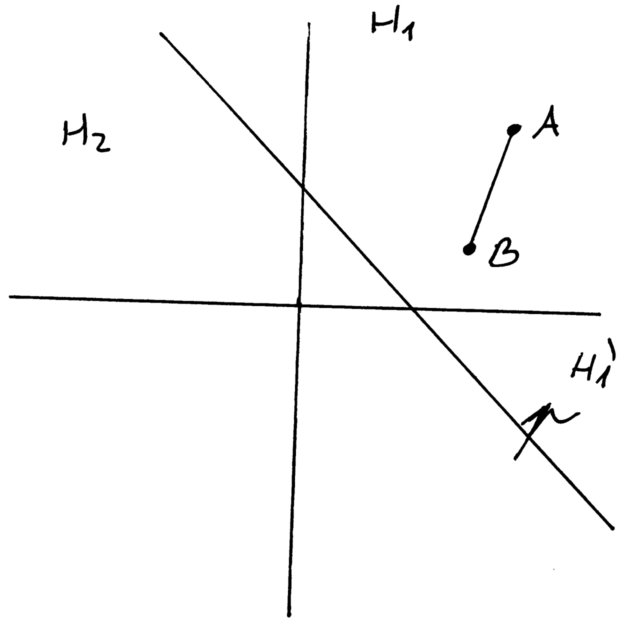
Metrička geometrija $\{\mathcal{S}, \mathcal{L}, d\}$ zadovoljava aksiom separacije (razdvajanja) ravni (PSA) ako za svaku pravu $p \in \mathcal{L}$ postoje dva podskupa H_1 i H_2 skupa $\mathcal{S}$ (koja nazivamo poluravni određene sa p) takve da

(i) $\mathcal{S} - p = H_1 \cup H_2$;

(ii) H_1 i H_2 su disjunktni i svaki je konveksan;

(iii) Ako je $A \in H_1$ i $B \in H_2$ tada $\overline{AB} \cap p \neq \emptyset$.

Teorema: Neka je p prava u metričnoj geometriji. Ako oba para H_1, H_2 i H_1', H_2' zadovoljavaju uslov PSA za pravu p tada ili $H_1 = H_1'$ (i $H_2 = H_2'$) ili $H_1 = H_2'$ (i $H_2 = H_1'$).



\# Dokazati teoremu iznad.

Rj. skica dokaza

$A \in H_1$

$A \notin p \Rightarrow A \in H_1'$ ili $A \in H_2'$

$A \in H_1'$, pokažimo $H_1 = H_1'$ (slično za $A \in H_2'$)

$H_1 \subseteq H_1'$:

$B \in H_1 \Rightarrow$ 1° $B = A \Rightarrow B \in H_1'$

2° $B \neq A \Rightarrow B \in H_1'$ ili $B \in H_2'$

$B \notin H_1' \Rightarrow B \in H_2'$ $(B \notin p)$

$\Downarrow$

$\overline{AB} \cap p \neq \emptyset$ $(A \in H_1', B \in H_2')$

$A \in H_1, B \in H_1$, H_1 konv. $\Rightarrow \overline{AB} \subseteq H_1 \Rightarrow \overline{AB} \cap p = \emptyset$

\# kontradikcija

Prema tome. $B \in H_1' \Rightarrow H_1 \subseteq H_1'$.

Slično $H_1' \subseteq H_1$.

Time $H_1 = H_1'$. $H_2 = \mathcal{S} - p - H_1 = \mathcal{S} - p - H_1' = H_2'$

$\Downarrow$

$H_2 = H_2'$

Definicija (pripadaju istoj strani, leže na suprotnim stranama, strana prave koja sadrži tačku)

Neka je $\{\mathcal{P}, \mathcal{L}, d\}$ metrička geometrija koja zadovoljava PSA, neka je $p \in \mathcal{L}$, i neka su H_1 i H_2 poluravni određene sa p. Za dvije tačke A i B kažemo da pripadaju istoj strani prave p ako obe pripadaju poluravni H_1 ili obe pripadaju poluravni H_2. Tačke A i B leže na suprotnim stranama prave p ako jedna od njih pripada poluravni H_1 a druga poluravni H_2. Ako je $A \in H_1$, kažemo da je H_1 strana prave p koja sadrži tačku A.

Teorem

Neka je $\{\mathcal{S}, \mathcal{L}, d\}$ metrična geometrija koja zadovoljava PSA. Neka su A i B dvije tačke iz $\mathcal{S}$ koje nisu na datoj pravoj ℓ. Tada

(i) A i B su sa suprotnih strana prave ℓ ako i samo ako $\overline{AB} \cap \ell \neq \emptyset$.

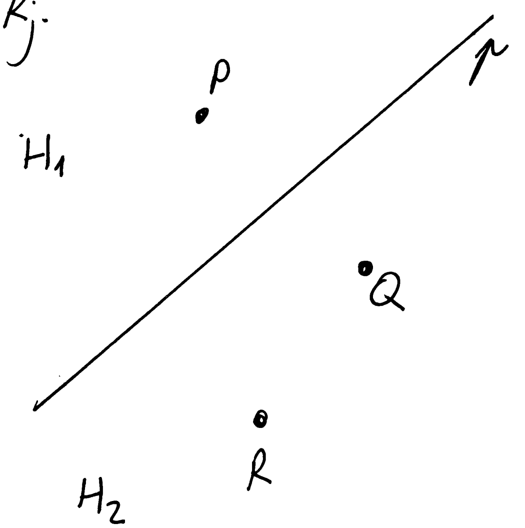
(ii) A i B su sa iste strane prave ℓ ako i samo ako je ili $A = B$ ili $\overline{AB} \cap \ell = \emptyset$.

\# Dokazati teoremu iznad.

Rj.

(#) Neka je p prava u metričnoj geometriji koja zadovoljava PSA. Ako su tačke P i Q sa različitih strana prave p i ako su Q i R sa iste strane prave p, pokazati da su tada P i R sa različitih strana prave p.

Rj:



$\{\mathcal{S}, \mathcal{L}, d\}$ zadovoljava PSA

$\Rightarrow \forall p \in \mathcal{L} \ \exists H_1, H_2 \subseteq \mathcal{S}$

- $\mathcal{S} - p = H_1 \cup H_2$
- H_1 i H_2 disjunktni, konveksni
- $A \in H_1,\ B \in H_2 \Rightarrow \overline{AB} \cap p \neq \emptyset$

P, Q sa različitih strana prave p

Pretp. da $P \in H_1$, $Q \in H_2$

Q i R su sa iste strane prave $p \xRightarrow{Q \in H_2} R \in H_2$

Time smo dobili da $P \in H_1$, $R \in H_2 \Rightarrow P$ i R su sa različitih strana prave p.

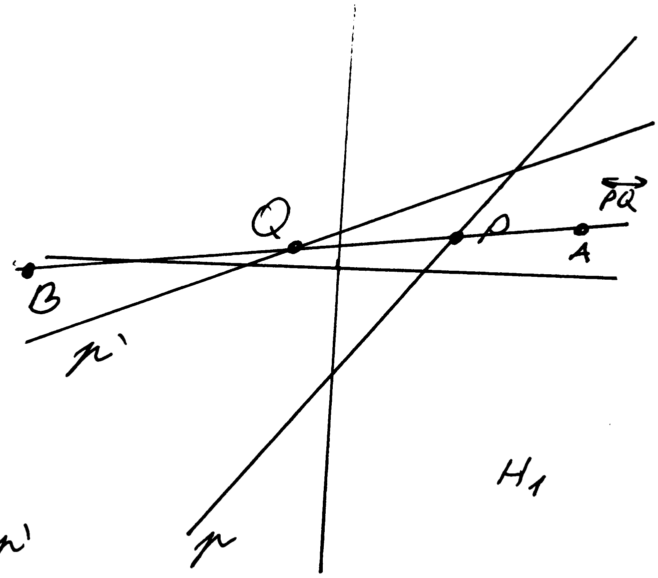
Definicija (ivica poluravni)

Ako je H_1 poluravan određena pravom p, tada je ivica poluravni H_1 prava p.

Teorema

Neka je p prava u metričkoj geometriji koja zadovoljava PSA. Pretpostavimo da je H_1 poluravan određena pravom p. Ako je H_1 također poluravan određena pravom p' tada je $p = p'$.

(#) Dokazati teoremu iznad.



Rj:

Skica dokaza

pretpostavimo $p \neq p'$. $\Rightarrow$ $p \cap p' \neq \emptyset$ ili $p \parallel p'$

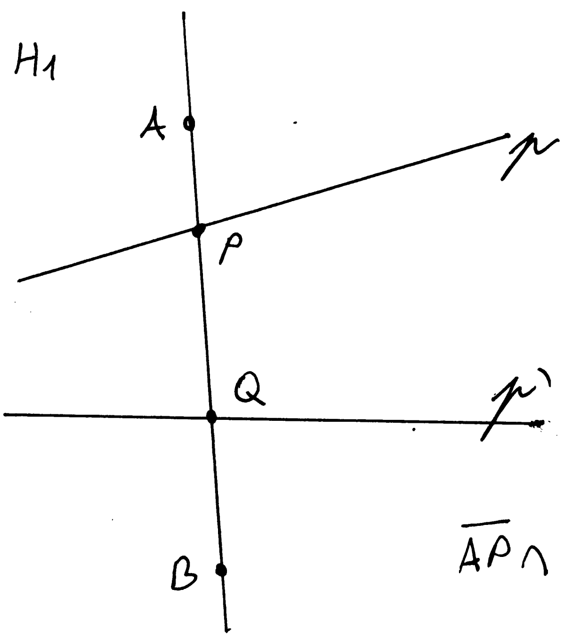
$\exists P$ t.d. $P \in p - p'$

$\exists Q$ t.d. $Q \in p' - p$

$P \notin p'$, $Q \notin p$ $\Rightarrow$ $\overleftrightarrow{PQ} \neq p$ i $\overleftrightarrow{PQ} \neq p'$

$\Rightarrow$ $\overleftrightarrow{PQ} \cap p = \{P\}$, $\overleftrightarrow{PQ} \cap p' = \{Q\}$

Izaberimo tačke $A, B \in \overleftrightarrow{PQ}$ t.d. $A-P-Q$, $P-Q-B$ $\Rightarrow$ $A-P-Q-B$



$$\left.\begin{array}{l} A-P-B,\ P \in \overline{AB} \cap p \\ \overline{AB} \cap p \subseteq \overleftrightarrow{PQ} \cap p = \{P\} \end{array}\right\} \Rightarrow \overline{AB} \cap p = \{P\}$$

$\Downarrow$ Teor.

A i B su sa suprotnih strana prave p.

A i B sa supr. str. pr. p $\Rightarrow$ $A \in H_1$ ili $B \in H_1$

1° $A \in H_1$

$$\left.\begin{array}{l} \overline{AP} \cap p' \subseteq \overleftrightarrow{PQ} \cap p' = \{Q\} \\ A-P-Q \end{array}\right\} \Rightarrow Q \notin AP \text{ i } \overline{AP} \cap p' = \emptyset.$$

$$\left.\begin{array}{l} A-P-Q \\ \overline{AP} \cap p' = \emptyset \end{array}\right\} \xRightarrow{\text{Teor.}}$$ A i P su sa iste strane prave p' i ta strana je poluravan H_1 s obzirom da je $A \in H_1$.

$$\Downarrow$$

$P \in H_1$

#kontradikaja

$(P \in p)$

2° $B \in H_1$

$$\left.\begin{array}{l} \overline{BQ} \cap p = \overleftrightarrow{PQ} \cap p = \{P\} \\ B-Q-P \end{array}\right\} \Rightarrow P \notin \overline{BQ} \quad \text{i} \quad \overline{BQ} \cap p = \emptyset$$

$$\Downarrow \text{Teor.}$$

B i Q su sa iste strane prave p i ta strana je poluravan H_1 (s obzirom da $B \in H_1$)

$$\Downarrow$$

$Q \in H_1$

#kontradikcija

$(Q \in p')$

Pretpostavka suprotna tvrdnji nas vodi u kontradikciju pa nije tačna. Prema tome $p = p'$.

11. Consider the distance function d_N defined on the Euclidean plane as follows: Let every line *other* than L_0 have the usual Euclidean ruler, and for the line L_0, let the ruler be $f : L_0 \to \mathbb{R}$ where

$$f((0,y)) = \begin{cases} y, & \text{if } y \text{ is not an integer,} \\ -y, & \text{if } y \text{ is an integer.} \end{cases}$$

(You may assume that this function is a ruler.)

(a) Show that $\{(0,y) \mid \frac{1}{2} \leqslant y \leqslant \frac{3}{2}\}$ is a convex set in $(\mathbb{R}^2, L_E, d_E)$, the Euclidean plane with the usual Euclidean distance, but not in $(\mathbb{R}^2, L_E, d_N)$, the Euclidean plane with the new distance.

(b) Find the line segment from $(0, \frac{1}{2})$ to $(0, \frac{3}{2})$ in $(\mathbb{R}^2, L_E, d_N)$. Show that it is a convex set in $(\mathbb{R}^2, L_E, d_N)$ but not in $(\mathbb{R}^2, L_E, d_E)$.

(c) Show that $(\mathbb{R}^2, L_E, d_N)$, the Euclidean plane with this new distance d_N, does *not* satisfy PSA, the Plane Separation Axiom.

SOLUTION:

(a) $\underline{\mathcal{S}_1 = \{(0,y) \mid \frac{1}{2} \leqslant y \leqslant \frac{3}{2}\} \text{ is convex in } (\mathbb{R}^2, \mathcal{L}_E, d_E):}$

Take $P, Q \in \mathcal{S}_1$, $P \neq Q$.
Without loss of generality say $P = (0, y_1)$, $Q = (0, y_2)$ and $y_1 < y_2$.

$$\begin{aligned} \text{Then} \quad \overline{PQ} &= \{P, Q\} \cup \{C \in \mathbb{R}^2 \mid P\text{—}C\text{—}Q\} \\ &= \{(0,y) \mid y_1 \leqslant y \leqslant y_2\}, \end{aligned}$$

in $(\mathbb{R}^2, \mathcal{L}_E, d_E)$.
Since $P, Q \in \mathcal{S}_1$, we have $\frac{1}{2} \leqslant y_1 < y_2 \leqslant \frac{3}{2}$, and so
$\overline{PQ} = \{(0,y) \mid y_1 \leqslant y \leqslant y_2\} \subseteq \{(0,y) \mid \frac{1}{2} \leqslant y \leqslant \frac{3}{2}\} = \mathcal{S}_1$.

Hence $\mathcal{S}_1$ is convex in $(\mathbb{R}^2, \mathcal{L}_E, d_E)$.

$\underline{\mathcal{S}_1 \text{ is not convex in } (\mathbb{R}^2, \mathcal{L}_E, d_N)}$:

Take $P = (0,1)$, $Q = (0, \frac{1}{2})$, so $P, Q \in \mathcal{S}_1$ and $P \neq Q$.

Let $C = (0,0)$. We claim that $C \in \overline{PQ}$ but $C \notin \mathcal{S}_1$, so $\mathcal{S}_1$ is not convex:

Now $f(P) = -1$, $f(Q) = \frac{1}{2}$ and $f(C) = 0$. And $P, C, Q \in L_0$ so these points are collinear.

Also $d_N(P,Q) + d_N(C,Q) = |f(P) - f(C)| + |f(C) - f(Q)| = 1 + \frac{1}{2} = \frac{3}{2}$.

And $d_N(P,Q) = |f(P) - f(Q)| = |-1 - \frac{1}{2}| = \frac{3}{2}$. Hence P—C—Q.
But $C = (0,0) \notin \mathcal{S}_1$. Hence $\mathcal{S}_1$ is NOT convex in $(\mathbb{R}^2, \mathcal{L}_E, d_N)$.

(b) Let $A = (0, \frac{1}{2})$ and $B = (0, \frac{3}{2})$. We want $\overline{AB}$ in $(\mathbb{R}^2, \mathcal{L}_E, d_N)$. Recall that $\overline{AB} = \{A, B\} \cup \{C \in \mathbb{R}^2 \mid A\text{—}C\text{—}B\}$.

Now 1 is the only integer between $\frac{1}{2}$ amd $\frac{3}{2}$, so in the geometry with the *new* metric d_N,

$$\overline{AB} \;=\; \{(0,y) \mid \frac{1}{2} \leqslant y \leqslant \frac{3}{2},\ y \neq 1\} \cup \{(0,-1)\}.$$

We shall now show that this is convex in $(\mathbb{R}^2, \mathcal{L}_E, d_N)$ but not in $(\mathbb{R}^2, \mathcal{L}_E, d_E)$.

Take $P, Q \in \overline{AB}$, $P \neq Q$; say $P = (0, y_1)$ and $Q = (0, y_2)$. We have the following cases:
(i) $\frac{1}{2} \leqslant y_1 < y_2 < 1$; (ii) $\frac{1}{2} \leqslant y_1 < 1 < y_2 \leqslant \frac{3}{2}$;
(iii) $1 < y_1 < y_2 \leqslant \frac{3}{2}$; (iv) $\frac{1}{2} \leqslant y_1 < 1,\ y_2 = -1$;
(v) $1 < y_2 \leqslant \frac{3}{2},\ y_1 = -1$.

In these five cases, $\overline{PQ}$ is, respectively,
(i) $\{(0,y) \mid y_1 \leqslant y \leqslant y_2\}$; (ii) $\{(0,y) \mid y_1 \leqslant y \leqslant y_2,\ y \neq 1\} \cup \{(0,-1)\}$;
(iii) same as (i); (iv) $\{(0,y) \mid y_1 \leqslant y < 1\} \cup \{(0,-1)\}$;
(v) $\{(0,y) \mid 1 < y \leqslant y_2\} \cup \{(0,-1)\}$.

In each case, $\overline{PQ} \subseteq \overline{AB}$, and so $\overline{AB}$ is indeed convex in $(\mathbb{R}^2, \mathcal{L}_E, d_N)$.

PSA za Euklidsku i Poincaré-ovu ravan

Oznake ($X^\perp$)

Ako je $X=(x,y)\in\mathbb{R}^2$ tada sa $X^\perp$ označavamo element

$$X^\perp=(-y,x)\in\mathbb{R}^2.$$

Lema

(a) Ako je $X \in \mathbb{R}^2$ tada je $\langle X, X^\perp \rangle = 0$.

(b) Ako je $X \in \mathbb{R}^2$ i $X \neq (0,0)$ tada $\langle Z, X^\perp \rangle = 0$ povlači da je $Z = tX$ za neko $t \in \mathbb{R}$.

(#) Dokazati lemu iznad.

Rj. Skica dokaza:

(a) $X = (x,y) \in \mathbb{R}^2 \Rightarrow X^\perp = (-y, x) \in \mathbb{R}^2$

$$\langle X, X^\perp \rangle = -xy + yx = 0$$

(b) $X = (x,y),\ Z = (z,w) \Rightarrow X^\perp = (-y,x),\ Z^\perp = (-w, z)$

$$\langle Z, X^\perp \rangle = -zy + wx \xRightarrow{\langle Z, X^\perp\rangle = 0} -zy + wx = 0 \quad \ldots (1)$$

$X \neq (0,0) \Rightarrow x \neq 0 \lor y \neq 0$

$x \neq 0 \xRightarrow{(1)} w = \frac{zy}{x} \Rightarrow Z = tX,\ t = \frac{z}{x}$

$y \neq 0 \xRightarrow{(1)} z = \frac{xw}{y} \Rightarrow Z = tX,\ t = \frac{w}{y}$

$Z = tX$ za neko $t \in \mathbb{R}$

Propozicija

Ako su P i Q različite tačke u $\mathbb{R}^2$ tada

$$\overleftrightarrow{PQ} = \{ A \in \mathbb{R}^2 \mid \langle A-P, (Q-P)^{\perp} \rangle = 0 \}$$

Dokazati propoziciju iznad.

Rj. Skica dokaza

$$\overleftrightarrow{PQ} \subseteq \{ A \in \mathbb{R}^2 \mid \langle A-P, (Q-P)^{\perp} \rangle = 0 \} \quad \ldots (1)$$

$A \in \overleftrightarrow{PQ} \Rightarrow A = P + t(Q-P)$, za neko $t \in \mathbb{R}$

$\langle A-P, (Q-P)^{\perp} \rangle = \langle t(Q-P), (Q-P)^{\perp} \rangle = t \langle Q-P, (Q-P)^{\perp} \rangle = 0$

$\Rightarrow A \in \{ A \in \mathbb{R}^2 \mid \langle A-P, (Q-P)^{\perp} \rangle = 0 \} \Rightarrow$ vrijedi (1).

$$\{ A \in \mathbb{R}^2 \mid \langle A-P, (Q-P)^{\perp} \rangle = 0 \} \subseteq \overleftrightarrow{PQ} \quad \ldots (2)$$

$A \in \mathbb{R}^2$, $A \in \{ M \in \mathbb{R}^2 \mid \langle M-P, (Q-P)^{\perp} \rangle = 0 \} \Rightarrow \langle A-P, (Q-P)^{\perp} \rangle = 0 \quad \ldots (**)$

$P \neq Q \Rightarrow Q-P \neq (0,0) \quad \ldots (***)$

$$\left.\begin{matrix} (**) \\ (***) \end{matrix}\right\} \xRightarrow{\text{preth. Lema}} \exists t \in \mathbb{R} \quad A-P = t(Q-P)$$

$$\Downarrow$$

$$A = P + t(Q-P) \in \overleftrightarrow{PQ}$$

$\Rightarrow$ vrijedi (2).

Na osnovu (1) i (2) tvrdnja slijedi.

Definicija (Euklidove poluravni)

Neka je $l = \overleftrightarrow{PQ}$ Euklidova prava. Euklidove poluravni određene sa l su

$$H^{+} = \{ A \in \mathbb{R}^2 \mid \langle (A-P), (Q-P)^{\perp} \rangle > 0 \}$$

$$H^{-} = \{ A \in \mathbb{R}^2 \mid \langle (A-P), (Q-P)^{\perp} \rangle < 0 \}$$

Propozicija

Euklidove poluravni određene sa $l=\overleftrightarrow{PQ}$ su konveksne.

(#) Dokazati propoziciju iznad.

Rj.

Skica dokaza

Posmatrajmo $H^+ = \{M \in \mathbb{R}^2 \mid \langle (M-P), (Q-P)^\perp \rangle > 0\}$

$A, B \in H^+ \Rightarrow \langle (A-P), (Q-P)^\perp \rangle > 0$

$\langle (B-P), (Q-P)^\perp \rangle > 0$

Pok. da $C \in \overline{AB} \Rightarrow C \in H^+$.

Kako $A, B \in H^+$, posm samo sluč. $A-C-B$

$$C \in \overline{AB} \overset{\text{Prop.}}{\Rightarrow} \exists t \; 0<t<1 \quad C = A + t(B-A) \Rightarrow C = (1-t)A + tB \quad 0<t<1$$

$$\Rightarrow \langle (C-P), (Q-P)^\perp \rangle = \langle ((1-t)A + tB - P), (Q-P)^\perp \rangle$$

$-P + tP - tP$

$$= \langle ((1-t)(A-P) + t(B-P)), (Q-P)^\perp \rangle$$

$$= (1-t)\underbrace{\langle (A-P), (Q-P)^\perp \rangle}_{>0} + t\underbrace{\langle (B-P), (Q-P)^\perp \rangle}_{>0} \quad \dots(1)$$

S obzirom da je $0<t<1$ to je i $(1-t)>0$ (kao i $t>0$) pa je (1) pozitivno. Time $\langle (C-P), (Q-P)^\perp \rangle > 0$, $C \in H^+$.

Slično za H^-.

Propozicija

Euklidova ravan zadovoljava PSA.

(#) Dokazati propoziciju iznad.

Rj.

Skica dokaza:

· $l = \overleftrightarrow{PQ}$

$A \in \mathbb{R}^2 \Rightarrow \langle (A-P), (Q-P)^\perp \rangle$ ili >0 ili $=0$ ili <0

$\Downarrow$ $A \in H^+$ $\Downarrow$ $A \in l$ $\Downarrow$ $A \in H^-$

$\Rightarrow \mathbb{R}^2 - l = H^+ \cup H^-$

Kako su H^+ i H^- disjunktni, a iz preth. Prop. i konveksni, ostalo je još da pok. da vrijedi uslov (iii) iz PSA.

$(A \in H_1, B \in H_2 \Rightarrow \overline{AB} \cap l \neq \emptyset)$

$A \in H^+, B \in H^-$ Odredimo t t.d. $0<t<1$ i $A + t(B-A) \in l$.

$$A + t(B-A) \in l \overset{\text{jedna od preth. Prop.}}{\Longleftrightarrow} \langle A + t(B-A) - P, (Q-P)^\perp \rangle = 0$$

$$\langle A-P, (Q-P)^\perp \rangle = -t \langle B-A, (Q-P)^\perp \rangle$$

$$= t \langle A-B, (Q-P)^\perp \rangle \quad \ldots (1)$$

$A \in H^+ \Rightarrow \langle A-P, (Q-P)^\perp \rangle > 0$

$$\langle A-B, (Q-P)^\perp \rangle = \left| A-B = (A-P)-(B-P) \right| = \underbrace{\langle A-P, (Q-P)^\perp \rangle}_{>0\ (A \in H^+)} - \underbrace{\langle B-P, (Q-P)^\perp \rangle}_{<0\ (B \in H^-)} \quad \ldots (2)$$

$\Rightarrow \langle A-B, (Q-P)^\perp \rangle > 0$

$$(1) \Rightarrow t = \frac{\langle (A-P), (Q-P)^\perp \rangle}{\langle (A-B), (Q-P)^\perp \rangle} > 0$$

$(2) \Rightarrow \langle A-B, (Q-P)^\perp \rangle > \langle A-P, (Q-P)^\perp \rangle$

$\Downarrow$

$t < 0$

Definicija

Ako je $\ell = {}_aL$ tip I prave u Poincaré-ovoj ravni tada Poincaré-ove poluravni određene sa ℓ su

$$H_+ = \{(x,y) \in \mathbb{H} \mid x > a\}$$

$$H_- = \{(x,y) \in \mathbb{H} \mid x < a\},$$

Ako je $\ell = {}_cL_r$ tip II prave tada Poincaré-ove poluravni određene sa ℓ su

$$H_+ = \{(x,y) \in \mathbb{H} \mid (x-c)^2 + y^2 > r^2\}$$

$$H_- = \{(x,y) \in \mathbb{H} \mid (x-c)^2 + y^2 < r^2\}$$

Propozicija

Poincaré-ova ravan zadovoljava PSA.

(#) Dokazati propoziciju iznad.

Rj. Skica dokaza

ℓ prava $\mathcal{H}$, H_+ i H_- Poincaré-ove poluravni odr. sa ℓ

$\overset{def.}{\Longrightarrow} \mathbb{H} - \ell = H_+ \cup H_-$; $H_+ \cap H_- = \emptyset$

Pokažimo da su poluravni konveksne i da je zadov. usl. (iii) iz definicije PSA.

$\ell = {}_cL_r$, $A, B \in \mathbb{H} - \ell$, $A \neq B$

parametriziramo duž $\overline{AB}$

$A(x_1, y_1)$, $B(x_2, y_2)$

1° $\overleftrightarrow{AB}$ tip I prave

2° $\overleftrightarrow{AB}$ tip II prave

1° pretp. da je $y_1 < y_2$

2° pretp da je $x_1 < x_2$

Bez obzira da li je 1° ili 2° pokazaćemo da $\exists$ f-ja $g(t)$ koja je ili uvijek rastuća ili uvijek opadajuća ili konstantna
Ova f-ja će biti $=0$ za tačke prave ℓ

1° $\overleftrightarrow{AB} = {}_aL \Rightarrow \overleftrightarrow{AB}$ parametr. $(x,y) \in \overleftrightarrow{AB}$ akko $(x,y) = (a, e^t)$ za neko $t \in \mathbb{R}$

Def. $f_1(t) = (a, e^t)$

f_1 je inverz. stand. mjere za ${}_aL$.

$$g_1(t) = (x-c)^2 + y^2 - r^2 = (a-c)^2 + e^{2t} - r^2$$

$g_1'(t) = 2e^{2t} > 0 \Rightarrow g_1$ uvijek raste.